HSEB Model Question - II (2068)

Mathematics

Grade: XII — Full Marks: 100

Time: 3 hrs — Pass Marks: 35

Candidates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt ALL questions of group A and group B or C.

Group A

1. a) **In an examination paper containing 10 questions, a candidate has to answer 7 questions. If two questions are made compulsory, in how many ways can he choose 7 questions in all?** **[2]**

Solution

If two questions are compulsory, then the candidate has to select 5 questions from the remaining 8 questions.

Therefore, he can choose 5 questions from 8 questions in

$$C(8, 5) = \frac{8!}{(8 - 5)!\ 5!} = \frac{8\ 7\ 6\ 5!}{3\ 2\ 1\ 5!} = 56 \text{ ways.}$$

b) **Find the middle term in the expansion of $\left(2x + \frac{1}{3x^2}\right)^9$** **[2]**

Solution

We have $n = 9$ which is odd. Hence there must be $9 + 1 = 10$ terms, which is even. This implies that there are two middle terms. They are

$$t_{(9-1)/2+1} = t_{4+1}$$

and $t_{(9-1)/2+1} = t_{5+1}$.

Now,

$$t_{4+1} = C(9, 4)\ (2x)^{9-4}\left(\frac{1}{3x^2}\right)^4$$

$$= \frac{9!}{(9-4)!\ 4!}\ (2x)^5\left(\frac{1}{3x^2}\right)^4$$

$$= \frac{9\ 8\ 7\ 6\ 5!}{5!\ 4\ 3\ 2\ 1}\ 2^5 \cdot x^5 \cdot \frac{1}{3^4} \cdot \frac{1}{x^8}$$

$$= \frac{448}{9}\ \frac{1}{x^3}.$$

Again

$$t_{5+1} = C(9, 5)\ (2x)^{9-5}\left(\frac{1}{3x^2}\right)^5$$

$$= \frac{9!}{(9-5)!\ 5!}\ (2x)^4\left(\frac{1}{3x^2}\right)^5$$

$$= \frac{9\ 8\ 7\ 6\ 5!}{5!\ 4\ 3\ 2\ 1}\ 2^4 \cdot x^4 \cdot \frac{1}{3^5} \cdot \frac{1}{x^{10}}$$

$$= \frac{224}{27}\ \frac{1}{x^6}.$$

c) **Let $S = \{-1, 1\}$ and $*$ denote the usual operation of multiplication. Represent it by Cayleyís table. Show that $*$ is a binary operation on S.** **[2]**

Solution

We have

$S = \{-1, 1\}$.

Cayley's table

*	−1	1
−1	1	−1
1	−1	1

From the table we observe that for any $x, y \in S$ the element $x * y = xy$ is unique and in S, so the multiplication $*$ is a binary operation on S.

2. a) **Find the eccentricity and the foci of the ellipse:**

$x^2 + 4y^2 - 4x + 24y + 24 = 0.$ **[2]**

Solution

We have

$$x^2 + 4y^2 - 4x + 24y + 24 = 0$$

$$\Rightarrow (x^2 - 4x) + 4(y^2 + 6y) = -24$$

$$\Rightarrow (x^2 - 4x + 4) - 4 + 4(y^2 + 6y + 9) - 9 = -24$$

$\Rightarrow \quad (x ñ 2)^2 + 4\,(y + 3)^2 = 16$

$\Rightarrow \quad \dfrac{(x ñ 2)^2}{16} + \dfrac{(y + 3)}{4} = 1.$

Comparing it with the equation $\dfrac{(x ñ h)^2}{a^2} + \dfrac{(y ñ k)^2}{b^2} = 1$, we get

$h = 2,\ k = ñ3,\ a^2 = 16$ and $b^2 = 4$.

Since $a > b$, so the major axis is parallel to x-axis. Then

Eccentricity $e = \sqrt{1 ñ \dfrac{b^2}{a^2}} = \sqrt{1 ñ \dfrac{4}{16}} = \sqrt{1 ñ \dfrac{1}{4}} = \dfrac{\sqrt{3}}{2}$.

Foci $= (h \pm ae, k) = (2 \pm 4 \cdot \dfrac{\sqrt{3}}{2}, ñ3) = (2 \pm 2\sqrt{3}, ñ3)$.

b) Find the point where the line through the points (1, 2, 3) and (4, ñ4, 9) meets the *zx*-plane. **[2]**

Solution

Let zx-plane divide the line joining the points (1, 2, 3) and (4, ñ4, 9) in the ratio $m_1 : m_2$. Then the co-ordinates of a point dividing the line in that ratio $m_1 : m_2$ is

$$\left(\frac{4m_1 + m_2}{m_1 + m_2}, \frac{ñ4m_1 + 2m_2}{m_1 + m_2}, \frac{9m_1 + 3m_2}{m_1 + m_2}\right)$$

If it lies in zx-plane, then its y-co-ordinate should be zero, so that

$$\frac{ñ4m_1 + 2m_2}{m_1 + m_2} = 0$$

$\Rightarrow \quad 4m_1 = 2m_2$

$\therefore \quad \dfrac{m_1}{m_2} = \dfrac{2}{4} = \dfrac{1}{2}$

Put $m_1 = 1$ and $m_2 = 2$, we get

$$\left(\frac{4 \cdot 1 + 2}{1 + 2}, \frac{ñ4 \cdot 1 + 2 \cdot 2}{1 + 2}, \frac{9 \cdot 1 + 3 \cdot 2}{1 + 2}\right) = (2, 0, 5),$$

which is the required point.

c) Are the three points with position vectors $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$ collinear? Justify your answer. **[2]**

Solution

Let O be the origin and three given points be A, B and C. Then

$\overrightarrow{OA} = \vec{i} + 2\vec{j} + 4\vec{k},\ \overrightarrow{OB} = 2\vec{i} + 5\vec{j} - \vec{k},\ \overrightarrow{OC} = 3\vec{i} + 8\vec{j} - 6\vec{k}$

Now,

$\overrightarrow{AB} = \overrightarrow{OB}\ ñ\ \overrightarrow{OA}$

$= (2\vec{i} + 5\vec{j}\ ñ\ \vec{k})\ ñ\ (\vec{i} + 2\vec{j} + 4\vec{k}) = \vec{i} + 3\vec{j}\ ñ\ 5\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC}\ ñ\ \overrightarrow{OA}$

$= (3\vec{i} + 8\vec{j}\ ñ\ 6\vec{k})\ ñ\ (\vec{i} + 2\vec{j} + 4\vec{k})$

$= 2\vec{i} + 6\vec{j}\ ñ\ 10\vec{k} = 2(\vec{i} + 3\vec{j}\ ñ\ 5\vec{k})$

$\therefore \quad \overrightarrow{AB} = 2\overrightarrow{AC}$

Therefore given points A, B and C are collinear since AC and AB are parallel.

We have

$\vec{a} = \vec{i} + 3\vec{j}\ ñ\ 7\vec{k}$

$\vec{b} = ñ5\vec{i} + 2\vec{j} + 4\vec{k}.$

If θ is the angle between $\vec{a}$ and $\vec{b}$, then the projection of $\vec{a}$ on $\vec{b}$ is

$$a \cos\theta = a\frac{\vec{a} \cdot \vec{b}}{ab} = \frac{\vec{a} \cdot \vec{b}}{b}$$

$$= \frac{ñ1\ \ 5 + 3\ \ 2\ ñ7\ \ 4}{\sqrt{(ñ5)^2 + 2^2 + 4^2}} = \frac{9}{\sqrt{5}}.$$

3. a) Using L'Hospitalís rule, evaluate

$$\lim_{x \to 0} \frac{e^x + e^{ñx}\ ñ\ 2\cos x}{\sin^2 x}.$$ **[2]**

Solution

$$\lim_{x \to 0} \frac{e^x + e^{ñx}\ ñ\ 2\cos x}{\sin^2 x} = \lim_{x \to 0} \frac{e^x\ ñ\ e^{ñx} + 2\sin x}{2\sin x \cos x}$$

$$= \lim_{x \to 0} \frac{e^x\ ñ\ e^{ñx} + 2\sin x}{\sin 2x}$$

$$= \lim_{x \to 0} \frac{e^x + e^{ñx} + 2\cos x}{2\cos 2x}$$

$$= \frac{1 + 1 + 2}{2\ \ 1} = 2.$$

b) Evaluate:

$$\int \frac{dx}{\sqrt{(x - \alpha)(x - \beta)}} \; (\beta > \alpha). \qquad [2]$$

Solution

$$\int \frac{dx}{\sqrt{(x - \alpha)(x - \beta)}} = \int \frac{1}{\sqrt{(x - \alpha)(x - \beta)}} \; \frac{\sqrt{x - \alpha} + \sqrt{x - \beta}}{\sqrt{x - \alpha} + \sqrt{x - \beta}} \, dx$$

$$= \int \left(\frac{1}{\sqrt{x - \beta}} + \frac{1}{\sqrt{x - \alpha}} \right) \frac{1}{\sqrt{x - \alpha} + \sqrt{x - \beta}} \, dx.$$

Put $y = \sqrt{x - \alpha} + \sqrt{x - \beta}$. Then

$$dy = \frac{1}{2}\left(\frac{1}{\sqrt{x - \alpha}} + \frac{1}{\sqrt{x - \beta}} \right) dx.$$

$$\therefore \int \frac{dx}{\sqrt{(x - \alpha)(x - \beta)}} = \int \frac{2}{y} \, dy = 2 \ln |y| + C$$

$$= 2 \ln |\sqrt{x - \alpha} + \sqrt{x - \beta}| + C.$$

c) If $\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$ show that $\vec{a} \quad \vec{b}$ is perpendicular to $\vec{a}$. **[2]**

Solution

We have

$$\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$$

$$\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$$

Now,

$$\vec{a} \quad \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -5 \\ 1 & -4 & 2 \end{vmatrix}$$

$$= \vec{i}\,(6 - 20) - \vec{j}\,(12 + 5) + \vec{k}\,(-24 - 3)$$

$$= -14\vec{i} - 17\vec{j} - 27\vec{k}.$$

Again,

$$(\vec{a} \quad \vec{b}) \cdot \vec{a} = (-14\vec{i} - 17\vec{j} - 27\vec{k}) \cdot (6\vec{i} + 3\vec{j} - 5\vec{k})$$

$$= -84 - 51 + 135 = 0.$$

$\therefore$ $(\vec{a} \quad \vec{b})$ is perpendicular to $\vec{a}$.

4. a) Solve: **[2]**

$$x\frac{dy}{dx} + y - 1 = 0. \qquad [2]$$

Solution

We have

$$x\frac{dy}{dx} + y - 1 = 0$$

$\Rightarrow$ $x\,dy + y\,dx - dx = 0$

$\Rightarrow$ $d\,(xy) - dx = 0.$

On integration we get

$xy - x = c$

$\Rightarrow$ $x\,(y - 1) = c.$

b) If $n = 10$, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$, find the correlation coefficient between the two variables. **[2]**

Solution

We have

$n = 10$, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$

We know that

$$\text{Correlation coefficient } (r) = \frac{n\,\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\,\Sigma X^2 - (\Sigma X)^2}\sqrt{n\,\Sigma Y^2 - (\Sigma Y)^2}}$$

$$= \frac{10 \quad 415 - 60 \quad 60}{\sqrt{10 \quad 400 - (60)^2}\sqrt{10 \quad 580 - (60)^2}}$$

$$= \frac{4150 - 3600}{\sqrt{400}\sqrt{2200}} = 0.59.$$

c) Two dice are rolled once. What is the probability of getting a total of 9 or 6? **[2]**

Solution

If two dices are rolled once, then the possible cases of turning up are

{(1, 1), (1, 2), (1, 3) . . . (2, 1), . . ., (2, 8), (3, 1), . . . (8, 8) }

There are 36 possible cases.

There are 4 cases having total of 9. They are

(3, 6), (4, 5), (5, 4), (6, 3).

There are 5 cases having total 6. They are

(1, 5), (2, 4), (3, 3), (4, 2), (5, 1).

$\therefore$ P (total of 9 or 6) $= P$(total of 9) + P (total of 6)

$$= \frac{4}{36} + \frac{5}{36} = \frac{9}{36} = \frac{1}{4}.$$

5. a) In how many ways can the letters of the word ìCOMPUTERî be arranged so that

i. all the vowels are always together?

ii. the vowels may occupy only odd positions? [4]

Solution

The word "COMPUTER' contains 8 letters among them, 'O', 'U' and 'E' are vowels.

i. If three vowels come together, then considering them to be one, there will be 6 letters to be arranged. Hence they can be arranged in P (6, 6) ways.

Again, three vowels can interchange their positions in P (3, 3) ways.

Therefore, by the multiplication principle of counting, the number of required arrangements in which the vowels are always come together is

P (6, 6) P(3, 3) = 6!   3! = 4320 ways.

ii. If the three vowels O, U, E occupy only in odd positions namely, 1^{st}, 3^{rd}, 5^{th} and 7^{th} positions i.e. 4 places, then the total number of arrangements of 3 vowels in 4 places in P(4, 3) ways.

Again, the remaining 5 letters can be arranged in remaining even 4 places in P(5, 4) ways.

Therefore, the number of required arrangements

$= P$ (5, 4) P (4, 3)

$= \frac{5!}{1!} \quad \frac{4!}{1!} = 5! \quad 4! = 120 \quad 24 = 2880$ ways.

b) Given the algebraic structure (G,*) with $G = \{1, \omega, \omega^2\}$ where ω represents an imaginary cube root of unity and * stands for the binary operation of multiplication, show that (G,*) is a group. [4]

Solution

Let $G = \{1, \omega, \omega^2\}$ where ω is the cube root of unity. Now, the set G with the operation of multiplication is a group. In fact,

i) Closure property:

$1\cdot\omega = \omega \in G, \quad \omega\cdot\omega^2 = \omega^3 = 1 \in G.$

$1\cdot\omega^2 = \omega^2 \in G$ and so on.

$\Rightarrow$ G is closed under multiplication.

ii) Associative property:

$1\cdot(\omega\cdot\omega^2) = 1\cdot\omega^3 = 1\cdot1 = 1$

$(1\cdot\omega)\cdot\omega^2 = \omega\cdot\omega^2 = \omega^3 = 1$

$\therefore$ $1(\omega\cdot\omega^2) = (1\cdot\omega)\cdot\omega^2$ for all $1, \omega, \omega^2 \in G$. Similarly for others.

iii) $1 \in G$ is a identity element, because

$1\cdot1 = 1,\ 1\cdot\omega = \omega\cdot1 = \omega, \quad 1\cdot\omega^2 = \omega^2\cdot1 = \omega^2.$

iv) Inverse element

We know that

$\omega\cdot\omega^2 = \omega^2\cdot\omega = \omega^3 = 1$, the identity element.

This implies that ω and ω^2 are inverse of each other. Also $1\cdot1 = 1$. Thus, 1 is inverse of itself.

Therefore G is a group with multiplication.

6. a) Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1). Express it in the slope form. [4]

OR

What is a conic section? Find the equation of the parabola in the standard form. [4]

Solution

We have

$y^2 = 4ax$

Differentiating both sides with respect to x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}.$$

Thus, the slope of the tangent to the parabola at a point (x_1, y_1) is

$$\frac{dy}{dx} = \frac{2a}{y_1}.$$

Therefore, the tangent to the parabola at (x_1, y_1) is given by the equation

$$y ñ y_1 = \frac{dy}{dx}(x ñ x_1)$$

$$\Rightarrow y ñ y_1 = \frac{2a}{y_1}(x ñ x_1)$$

$$\Rightarrow yy_1 ñ y_1^2 = 2ax ñ 2ax_1$$

$$\Rightarrow yy_1 ñ 4ax_1 = 2ax ñ 2ax_1$$

$$\Rightarrow yy_1 = 2a(x + x_1). \quad [\because y_1^2 = 4ax_1]$$

Second Part

The line $y = mx + c$ will be tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

Therefore, the equation of the tangent to the parabola in the slope form is

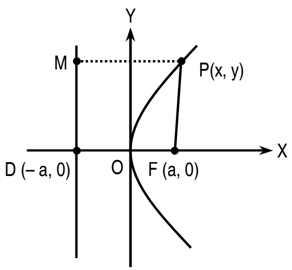
$$y = mx + \frac{a}{m}.$$

OR Part

Conic Section: A conic section is defined as the locus of a point which moves in the plane so that the ratio of its distance from a fixed point to its distance from a fixed straight line remains the same throughout the motion.

Second Part

In order to obtain the equation of a parabola in a standard form we consider a parabola with vertex at the origin and focus at $F(a, 0)$, where $a > 0$. It is clear that the axis of the parabola is the x-axis and the directrix of the parabola is parallel to the y-axis. If the directrix intersects the x-axis at D, then by definition, we must have



$$OF = OD.$$

This indicates that F and D must lie on the opposite sides of the origin O. Thus, D has coordinates $(ñ a, 0)$. Hence the directrix is given by the equation

$$x = ñ a$$

i.e. $x + a = 0$.

Let $P(x, y)$ be any point on the parabola. Draw PF and perpendicular PM meeting the directrix at M.

Clearly, M has coordinates $(ñ a, y)$. By definition we have

$$PF = PM$$

$$\Rightarrow PF^2 = PM^2$$

$$\Rightarrow (x ñ a)^2 + (y ñ 0)^2 = (x + a)^2 + (y ñ y)^2$$

$$\Rightarrow (x ñ a)^2 + y^2 = (x + a)^2.$$

$$\therefore y^2 = 4ax.$$

b) Find the equation of the plane through the point (2, 1, 4) and perpendicular to each of the planes

$9x - 7y + 6z + 48 = 0$ and $x + y + z = 0$. **[4]**

Solution

A plane through the point (2, 1, 4) is given by the equation

$$a(x ñ 2) + b(y ñ 1) + c(z - 4) = 0 \quad \ldots (i)$$

If it is perpendicular to both planes

$$9x ñ 7y + 6z + 48 = 0 \text{ and } x + y + z = 0$$

then we must have

$$9a ñ 7b + 6c = 0 \quad \ldots (ii)$$

$$a + b + c = 0 \quad \ldots (iii)$$

By cross multiplication method, we get

$$\frac{a}{ñ7 ñ 6} = \frac{b}{6 ñ 9} = \frac{c}{9 + 7} = K \text{ (suppose)}$$

$$\Rightarrow a = ñ13K, b = ñ3K \text{ and } c = 16K$$

Putting these values in (i), we get

$$ñ13K(x ñ 2) ñ 3K(y ñ 1) + 16K(z - 4) = 0 \quad \ldots (i)$$

$$\Rightarrow 13x ñ 26 + 3y ñ 3 ñ 16z + 64 = 0$$

$$\Rightarrow 13x + 3y ñ 16z + 35 = 0.$$

7. a) Evaluate:

$$\int \frac{dx}{a + b\cos x} \quad (a > b > 0).$$ **[4]**

Solution

Put $y = \tan\frac{x}{2}$. Then

$$dy = \frac{1}{2}\sec^2\frac{x}{2}\,dx, \text{ i.e. } dx = \frac{2}{\sec^2(x/2)}\,dy = \frac{2}{y^2+1}\,dy.$$

Also, $\cos x = \frac{1 ñ y^2}{1+y^2}$.

We have

$$I = \int \frac{dx}{a + b\cos x} = \int \frac{1}{a + b\,\frac{1 ñ y^2}{1+y^2}} \quad \frac{2}{y^2+1}\,dy$$

$$= 2\int \frac{1}{(a+b) + (a ñ b)y^2}\,dy.$$

If $a > b > 0$, then

$$c^2 = \frac{a+b}{a ñ b} > 0 \text{ and so } c = \sqrt{\frac{a+b}{a ñ b}}.$$

Now,

$$I = \frac{2}{a ñ b}\int \frac{1}{c^2+y^2}\,dy = \frac{2}{a ñ b} \quad \frac{1}{c}\tan^{ñ1}\frac{y}{c} + C$$

$$= \frac{a ñ b}{|a ñ b|} \quad \frac{2}{\sqrt{a^2 ñ b^2}}\tan^{ñ1}\left(\sqrt{\frac{a ñ b}{a+b}}\tan\frac{x}{2}\right) + C.$$

b) Solve: $x^2\frac{dy}{dx} + y^2 = xy$. **[4]**

OR

Solve: $(1 ñ x^2)\frac{dy}{dx}$ ñ xy = 1 **[4]**

Solution

We have

$$x^2\frac{dy}{dx} + y^2 = xy$$

$$\Rightarrow \frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}. \qquad \text{. . . (i)}$$

This is a homogeneous differential equation. So put $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Hence equation (i) becomes

$$v + x\frac{dv}{dx} + \frac{v^2x^2}{x^2} = \frac{vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} + v^2 = v$$

$$\Rightarrow x\frac{dv}{dx} = ñ v^2$$

$$\Rightarrow ñ\frac{dx}{x} = v^{ñ2}\,dv$$

On integration, we get

$$\frac{v^{ñ1}}{ñ1} = \ln x ñ C$$

$$\Rightarrow \frac{1}{v} = ñ\ln x + C$$

$$\Rightarrow \frac{x}{y} = ñ\ln x + C$$

$$\therefore x + y\ln x = yc.$$

OR Part

We have

$$(1 ñ x^2)\frac{dy}{dx} ñ xy = 1$$

$$\Rightarrow \frac{dy}{dx} ñ \frac{xy}{1 ñ x^2} = \frac{1}{1 ñ x^2} \qquad \text{. . . (i)}$$

Comparing it with $\frac{dy}{dx} + Py = Q$ we get

$$P = \frac{ñ x}{1 ñ x^2},\ Q = \frac{1}{1 ñ x^2}$$

$$\text{I.F.} = e^{\int P dx} = e^{ñ\int x/1 ñ x^2\,dx} = e^{1/2\ln(1 ñ x^2)} = e^{\ln\sqrt{1-x^2}} = \sqrt{1 ñ x^2}$$

Multiplying (i) by I.F., we get

$$\sqrt{1 ñ x^2}\,\frac{dy}{dx} ñ \frac{x}{\sqrt{1 ñ x^2}}\,y = \frac{1}{\sqrt{1 ñ x^2}}$$

$$\Rightarrow \frac{d}{dx}\left(\sqrt{1 ñ x^2}\,y\right) = \frac{1}{\sqrt{1 ñ x^2}}$$

On integration, we get

$$\sqrt{1 ñ x^2}\,y = \int \frac{1}{\sqrt{1 ñ x^2}}\,dx$$

$$\therefore y\sqrt{1 ñ x^2} = \sin^{ñ1} x + C.$$

8. a) Find Karl Pearsonís coefficient of skewness from the following distribution. **[4]**

Marks	Above 20	Above 30	Above 40	Above 50	Above 60
No. of students	50	46	30	24	8

Solution

Calculation of Karl Pearson's coefficient of skewness

Marks	Mid point (X)	f	c.f.	$d' = \frac{x - A}{10}$	fd'	fd'^2
20 – 30	25	4	4	–2	–8	16
30 – 40	35	16	20	–1	–16	16
40 – 50	45	6	26	0	0	0
50 – 60	55	16	42	1	16	16
60 – 70	65	8	50	2	16	32
		$N = 50$			$\Sigma fd' = 8$	$\Sigma fd'^2 = 80$

For mean

$$\bar{X} = A + \frac{\Sigma fd'}{N} \quad i = 45 + \frac{8}{50} \quad 10 = 46.6.$$

For median

$$M_d = \left(\frac{N}{2}\right)^{th} \text{ item} = \left(\frac{50}{2}\right)^{th} \text{ item} = 25 \text{ item}$$

So median lies in the class 40 – 50, we get

$$M_d = l + \frac{\frac{N}{2} - \text{c.f.}}{f} \quad i = 40 + \frac{25 - 20}{6} \quad 10 = 48.33$$

Now,

$$\sigma = \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \quad i$$

$$= \sqrt{\frac{80}{50} - \left(\frac{8}{50}\right)^2} \quad 10 = \sqrt{1.6 - 0.0256} \quad 10 = 12.55$$

$$\therefore S_K(P) = \frac{3(\bar{X} - M_d)}{\sigma} = \frac{3(46.6 - 48.33)}{12.55} = -0.41$$

b) The chance that A can solve a certain problem is $\frac{1}{4}$ and the chance that B can solve it is $\frac{2}{3}$. Find the chance that (i) the problem will be solved if they both try (ii) A solves but B cannot.

OR

Suppose that in a certain city 60% of all the recorded births are male. Suppose we select 5 birth records from population. What is the probability that

i. exactly three of them are male?

ii. 4 or more are male? **[4]**

Solution

We have

$$P(A) = \frac{1}{4} \qquad P(B) = \frac{2}{3}.$$

i. $P(A \text{ or } B) = ?$

We know that

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

$$= \frac{1}{4} + \frac{2}{3} - \frac{1}{4} \quad \frac{2}{3} = \frac{9}{12} = \frac{3}{4}$$

ii. $P(\bar{B}) = 1 - \frac{2}{3} = \frac{1}{3}$

$$P(A \cap \bar{B}) = P(A)\, P(\bar{B}) = \frac{1}{4} \quad \frac{1}{3} = \frac{1}{12}.$$

OR Part

We have

$$p = 60\% = \frac{60}{100} = \frac{3}{5}, \qquad q = 1 - \frac{3}{5} = \frac{2}{5}$$

Number of trials (n) = 5

Now, probability of r successes in 5 trails is given by

$$p(r) = C(5, r)\,(p)^r\,(q)^{5-r} = C(5, r)\left(\frac{3}{5}\right)^r \left(\frac{2}{5}\right)^{5-r}$$

i. $P(\text{Exactly three male}) = P(3)$

$$= C(5, 3)\left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^{5-3}$$

$$= \frac{5!}{(5-3)!\,3!} \quad \frac{27}{125} \quad \frac{4}{25}$$

$$= \frac{5 \quad 4 \quad 3!}{2! \quad 3!} \quad \frac{27}{125} \quad \frac{4}{25}$$

$$= \frac{216}{625}.$$

ii. $P(r \ge 4) = P(5) = C(5, 5)\left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^{5-5} = 1 \quad \frac{3^5}{5^9} = \frac{343}{3125}.$

9. Show that

$$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2} = e - 1. \qquad [6]$$

Solution

We have

$$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2} = \sum_{n=2}^{\infty} \frac{(n-1)^2}{n!} = \sum_{n=2}^{\infty} \frac{n^2}{n!} - 2\sum_{n=2}^{\infty} \frac{n}{n!} + \sum_{n=2}^{\infty} \frac{1}{n!}$$

$$= \left(\frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots\right) - 2\left(\frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \ldots\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\right)$$

$$= \left(\frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \ldots\right) - 2\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + \left(-2 + 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\right)$$

$$= \left(\frac{1+1}{1!} + \frac{2+1}{2!} + \frac{3+1}{3!} + \ldots\right) - 2\left(-1 + 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + (e - 2)$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) - 2(e - 1) + (e - 2)$$

$$= \left(-1 + 2 + \frac{2}{1!} + \frac{2}{2!} + \frac{2}{3!} + \ldots\right) - 2(e - 1) + (e - 2)$$

$$= \left\{2\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) - 1\right\} - 2(e - 1) + (e - 2)$$

$$= 2e - 1 - 2e + 2 + e - 2$$

$$= e - 1.$$

10. Define scalar product of two vectors. Find the geometrical interpretation of scalar product of two vectors. Prove vectorically that

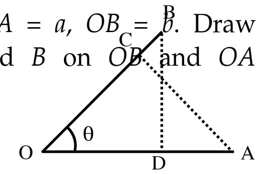
$\cos(A + B) = \cos A \cos B - \sin A \sin B$ **[6]**

Solution

Scalar Product of two vectors: Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be any two vectors in space. The scalar product or dot product of vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Geometric Interpretation: Let θ be the angle between two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ so that $OA = a$, $OB = b$. Draw perpendiculars AC and BD from points A and B on OB and OA respectively.



Now,

$$\vec{a} \cdot \vec{b} = ab \cos\theta = (OA)(OB)\cos\theta$$
$$= (OA)(OB\cos\theta) = (OA)(OD)$$
$$= \text{(Length of } OA\text{) (Projection of } OB \text{ on } OA\text{)}$$

Similarly,

$$\vec{a} \cdot \vec{b} = ab \cos\theta = (OA)(OB)\cos\theta$$
$$= (OB)(OA\cos\theta) = (OB)(OC)$$
$$= \text{(Length of } OB\text{) (Projection of } OA \text{ on } OB\text{)}$$

Thus, geometrically the dot product $\vec{a} \cdot \vec{b}$ is the product of the length of either vector and the projection of the other onto its direction.

Last Part

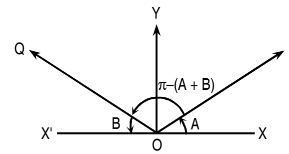
Let O be the origin and OX and OY be x-axis and y-axis respectively.

Let $\angle XOP = A$, $\angle X'OQ = B$.

So that $\angle POQ = \pi - (A + B)$

Let $OP = r_1$ and $OQ = r_2$. Then

Co-ordinates of P & Q are $(r_1 \cos A, r_1 \sin A)$ and $(-r_2 \cos B, r_2 \sin B)$ respectively.



$\therefore \quad \overrightarrow{OP} = (r_1 \cos A, r_1 \sin A)$

$\overrightarrow{OQ} = (-r_2 \cos B, r_2 \sin B)$

Now,

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = (r_1 \cos A, r_1 \sin A)(-r_2 \cos B, r_2 \sin B)$$
$$= r_1 r_2 (-\cos A \cos B + \sin A \sin B)$$
$$= -r_1 r_2 (\cos A \cos B - \sin A \sin B)$$

By definition

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}|\ |\overrightarrow{OQ}| \cos(\pi - (A + B))$$

$\Rightarrow \quad -r_1 r_2 (\cos A \cos B - \sin A \sin B) = -r_1 r_2 \cos(A + B)$

$\therefore \quad \cos(A + B) = \cos A \cos B - \sin A \sin B.$

11. State Rolleís theorem. Interpret it geometrically. Verify Rolleís theorem for the function

$$f(x) = x\,(x - 1)^2 \text{ in } [0,1]$$

Also, find the point on the curve where the tangent is parallel to the x-axis. [6]

OR

Find from first principle the derivative of $\ln \cos^{ñ1} x$. [6]

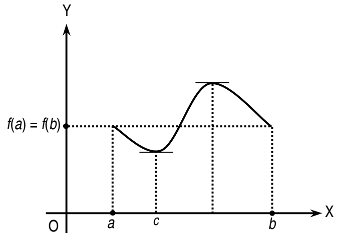
Solution

Rolle's Theorem

Let f be continuous on [a, b] and differentiable on (a, b). If f (a) = f (b), then

$$\exists\, c \in (a, b) : f\,'(c) = 0$$

If a function on $[a, b]$ satisfies the conditions of Rolle's Theorem, then the theorem geometrically indicates that there is a horizontal tangent at a point $c \in (a, b)$.



Second Part

We have

$f(x) = x\,(x ñ 1)^2 = x^3 ñ 2x^2 + x$

We observe that for any $p \in [0, 2]$, the value

$f(p) = p^3 ñ 3p^2 + p$

is a real number and $\lim\limits_{x\to p} f(x) = p^3 ñ 3p^2 + p$.

Hence $f(x)$ is continuous on [ñ1, 1].

We also have

$f\,'(x) = 3x^2 ñ 4x + 1$

It is defined for all $n \in (0, 1)$. Hence f is differentiable on (0, 1). Moreover, $f(0) = f(1) = 0$.

Therefore, the function on [0, 1] satisfies all the conditions of Rolle's Theorem. Now, by Rolle's Theorem, there is a number $c \in$ (ñ0, 1) such that

$f\,'(c) = 0$

$\Leftrightarrow\ 3c^2 ñ 4c + 1 = 0$

$\Leftrightarrow\ 3c^2 ñ 3c ñ c + 1 = 0$

$\Leftrightarrow\ 3c\,(c - 1)\ ñ 1\,(c ñ 1) = 0$

$\Leftrightarrow\ (c ñ 1)\ (3c ñ 1) = 0$

$\Leftrightarrow\ c = 1, 1/3$

Clearly, $c = 1, \frac{1}{3} \in (0, 1)$

Hence Rolle's Theorem is verified.

At $c = 1, \frac{1}{3}$, the tangent is parallel to the x-axis.

OR Part

Let $f(x) = \ln \cos^{ñ1} x$. Then, $f(x + h) = \ln \cos^{ñ1}(x + h)$

We have

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x + h) ñ f(x)}{h} = \lim_{h\to 0} \frac{\ln \cos^{ñ1}(x + h) ñ \ln \cos^{ñ1} x}{h}$$

Put $\cos^{ñ1} x = y$ and $\cos^{ñ1}(x + h) = y + k$.

Then, $x = \cos y$, $x + h = \cos(y + k)$.

Also, $h = \cos(y + k) ñ \cos y$ and $k \to 0$ as $h \to 0$

Now,

$$\frac{d}{dx} f(x) = \lim_{k\to 0} \frac{\ln(y + k) ñ \ln y}{k} \quad \lim_{k\to 0} \frac{k}{h}$$

$$= \lim_{k\to 0} \frac{\ln\left(1 + \frac{k}{y}\right)}{y \quad \frac{k}{y}} \quad \lim_{k\to 0} \frac{k}{\cos(y + k) ñ \cos y}$$

$$= \frac{1}{y} \quad 1 \quad \lim_{k\to 0} \frac{k}{ñ2\sin\left(\frac{2y + k}{2}\right) \sin\frac{k}{2}}$$

$$= \frac{ñ1}{\cos^{ñ1} x} \lim_{k\to 0} \frac{k/2}{\sin\frac{k}{2}} \quad \frac{1}{\sin\left(\frac{2y + k}{2}\right)}$$

$$= \frac{ñ1}{\cos^{ñ1} x} \quad \frac{1}{\sin y} = \frac{ñ1}{\cos^{ñ1} x \sqrt{1 ñ \cos^2 y}} = ñ\frac{1}{\cos^{ñ1} x\,(\sqrt{1 ñ x^2})}$$

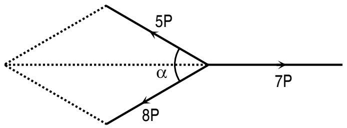
$$\therefore \quad \frac{d \ln \cos^{ñ1} x}{dx} = \frac{ñ1}{\sqrt{1 ñ x^2}\,\cos^{ñ1} x}$$

Group ëBí

12. a) Forces equal to 7P, 5P and 8P acting on a particle are in equilibrium. Find the angle between the latter pair of forces. [2]

Solution

Since the given forces are in equilibrium, the resultant of the latter pair of forces: $5P$ and $8P$ is equal in magnitude and opposite in direction to $7P$. Hence if α is the angle between the forces $5P$ and $8P$, then



$7^2 = 5^2 + 8^2 + 2\ \ 5\ \ 8 \cos\alpha$

$\Rightarrow \cos\alpha = ñ1/2$

$\therefore \alpha = 120°$.

b) A body is projected vertically upwards with a velocity of 19.6 m/s. How long will it take to reach a point 294m below the point of projection? (g = 9.8m/s) [2]

Solution

We have

Initial velocity (u) = 19.6 m/s

Height (h) = 294 m

Time (t) = ?

We know that

$ñh = ut\ ñ\frac{1}{2}gt^2$ (taking upward motion as positive)

$\Rightarrow ñ294 = 19.6\,t\ ñ\frac{1}{2}\ \ 9.8\ \ t^2$

$\Rightarrow 4.9t^2\ ñ\ 19.6\,t\ ñ\ 294 = 0$

$\Rightarrow t = \frac{19.6 \pm \sqrt{(ñ19.6)^2\ ñ\ 4\ \ 4.9\ \ (ñ294)}}{2\ \ 4.9}$

$= \frac{19.6 \pm \sqrt{384.16 + 5762.4}}{9.8}$

$= \frac{19.6 \pm \sqrt{6146.556}}{9.8}$

$= \frac{19.6 \pm 78.4}{9.8}$

Since time is not negative, so we get

$t = \frac{98}{9.8} = 10$ sec.

c) A body of mass 50 kg falling from a certain height is brought to rest after striking the ground with a speed of 5 m/s. If the resistance force of the ground is 500N, find the duration of the contact. [2]

Solution

We have

Mass (m) = 50 kg

Initial velocity (u) = $5ms^{ñ1}$

Final velocity (v) = 0

Force (F) = 500 N

Duration of contact (t) = ?

We know that

$F = \frac{m\,(v\ ñ\ u)}{t}$

$\Rightarrow t = \frac{m\,(v\ ñ\ u)}{F} = \frac{50\,(0\ ñ\ 5)}{500} = \frac{ñ250}{500} = ñ0.5$

$\therefore t = 0.5$ sec.

13. a) P and Q are two like parallel forces acting at A and B. Show that if they interchange positions, the point of application of the resultant is displaced by a distance $\frac{P\ ñ\ Q}{P + Q} \cdot AB$. [4]

OR

Forces 1N, 2N and 3N act at a point in direction parallel to the sides of an equilateral triangle taken in order. Find their resultant. [4]

Solution

Let P and Q be two like parallel forces acting at points A and B so that their resultant $P + Q$ act at point C, as shown in fig. (a). By the theorem on parallel forces, we have

$\frac{P}{BC} = \frac{Q}{AC} = \frac{P + Q}{AB}$

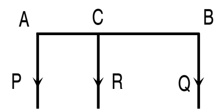


Fig (a)

i.e. $AC = \frac{Q}{P + Q} AB$ . . . (1)

If forces be interchanged in their position, then let its resultant act at point C_1, as shown in fig. (b). Then

$$\frac{P}{AC_1} = \frac{Q}{BC_1} = \frac{P+Q}{AB}$$

i.e. $AC_1 = \dfrac{P.\,AB}{P+Q}$ . . . (2)

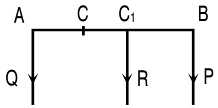


Fig (b)

Now, $AC_1 - AC = \dfrac{P - Q}{P+Q}\,AB$

Hence the line of action of the resultant is displaced along AB through a distance,

$$d = \frac{P - Q}{P+Q}AB, \text{ where } P > Q.$$

OR Part

Let three forces P, $2P$ and $3P$ along the sides AB, BC and CA respectively of equilateral triangle. Then

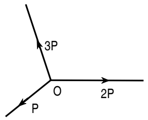
$R_x = 2P \cos 0 + 3P \cos 120 + P \cos 240$

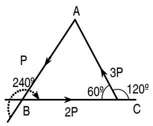
$= 2P - 3P \;\; \dfrac{1}{2} - P \;\; \dfrac{1}{2} = \dfrac{4P - 3P - P}{2} = 0$

$R_y = 2P \sin 0 + 3P \sin 120 + P \sin 240$

$= 0 + 3P \;\; \dfrac{\sqrt{3}}{2} - P \;\; \dfrac{\sqrt{3}}{2}$

$= \dfrac{2\sqrt{3}}{2} P = \sqrt{3}\, P$





Therefore, the resultant is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{0 + \sqrt{3}^2 P^2} = \sqrt{3}\, P$$

For the direction of resultant,

$$\cot\theta = \frac{R_x}{R_y} = \frac{0}{\sqrt{3}\,P} = 0.$$

$\therefore \quad \theta = 90 \qquad [\because R_y > 0]$

Hence the magnitude of the resultant is $\sqrt{3}\,P$ at right angle to the force $2P$.

b) Prove that the sum of the kinetic and potential energies of a freely falling body remains constant throughout the motion. [4]

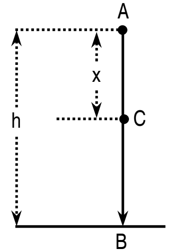
Solution

Let a body of mass be at point A which is at a height h above the ground, B the point where it reaches the ground and C any point on the way including A and B. If $AC = x$ is the height the body falls, then

the body acquires some velocity v. Since its initial velocity is zero and the acceleration due to gravity is g, we have

$v^2 = 2gx$

so that



$K.E. = \dfrac{1}{2}mv^2 = mgx$

Also, point C is at a height $h - x$ above the ground so that

$P.E. = mg(h - x)$

$\therefore \quad K.E. + P.E. = mgx + mg(h - x) = mgh$

We see that mgh is independent of x, the height the body falls. Therefore, the sum of kinetic and potential energies of a body being mgh, remains constant throughout free fall.

Notice that at start i.e. at point A, $K.E.$ is zero, since the initial velocity is zero but at A, $P.E$ is mgh.

$\therefore \quad K.\,E. + P.E. = mgh$

If v is the final velocity acquired by the body on reaching the ground i.e. at B. Then

$v^2 = 2gh$

$K.E. \text{ at } B = \dfrac{1}{2}m \quad 2gh = mgh.$

But at B, $P.E.$ is zero since at B height is 0.

$\therefore \quad K.E. + P.E. = mgh.$

Thus sum of kinetic and potential energies of a freely falling body remains constant throughout the motion.

14. The horizontal and the vertical components of the initial velocity of a projectile are U and V respectively. If R be the horizontal range and H, the greatest height attained, prove that

i) $\dfrac{4H}{R} = \dfrac{V}{U}$. **ii)** $\left(\dfrac{R}{U}\right)^2 = \dfrac{8H}{g}$. **[6]**

OR

A cat seeing a mouse at a distance of 15m before it, starts from rest with an acceleration of 2 m/s² and pursues it. If the mouse be moving uniformly with a velocity of 14 m/s, find when and where the cat will catch the mouse. [6]

Solution

i. Let u = velocity of projection, α = angle of projection. Then

Horizontal component of $u = u \cos \alpha = U$

Vertical component of $u = u \sin \alpha = V$

and Range $= R = \dfrac{u^2 \sin 2\alpha}{g}$ and greatest height $= H = \dfrac{u^2 \sin^2 \alpha}{2g}$

Now,

$$\frac{4H}{R} = \frac{4\dfrac{u^2 \sin^2 \alpha}{2g}}{\dfrac{u^2 \sin 2\alpha}{g}}$$

$$= \frac{2\sin^2 \alpha}{\sin 2\alpha} = \frac{2\sin^2 \alpha}{2\sin \alpha \cos \alpha} = \frac{\sin \alpha}{\cos \alpha} = \frac{u \sin \alpha}{v \cos \alpha} = \frac{V}{U}.$$

ii. $$\left(\frac{R}{U}\right)^2 = \left(\frac{\dfrac{u^2 \sin 2\alpha}{2g}}{u \cos \alpha}\right)^2 = \left(\frac{u^2 \sin 2\alpha}{g} \quad \frac{1}{u \cos \alpha}\right)^2$$

$$= \left(\frac{2u^2 \sin \alpha \cos \alpha}{g\, u \cos \alpha}\right)^2 = \frac{4u^2 \sin^2 \alpha}{g^2} = \frac{8u^2 \sin^2 \alpha}{2g} \quad \frac{1}{g} = \frac{8H}{g}$$

<u>**OR Part**</u>

Let t be the time taken by the cat to catch the mouse and let the mouse run x m in t sec. Then the distance covered by the cat to catch the mouse is 15 m + x m.

Since the mouse runs with uniform velocity, using $s = ut$, we get

$x = 14t$ . . . (i)

For the cat, using $s = ut + \frac{1}{2}at^2$, we get

$15 + x = 0 + \frac{1}{2} \quad 2 \quad t^2,$

$\Rightarrow$ $t^2 = 15 + x$ . . . (ii)

From (i) and (ii), we get

$t^2 = 15 + 14t$

$\Rightarrow$ t^2 ñ $14t$ ñ 15 = 0

$\Rightarrow$ t^2 ñ $15t + t$ ñ 15 = 0

$\Rightarrow$ t (t ñ 15) + 1 (t ñ 15) = 0

$\Rightarrow$ (t + 1) (t ñ 15) = 0

$\Rightarrow$ t = ñ 1 or 15.

$\therefore$ t = 15 sec [$\because$ t = ñ1 sec is impossible]

Then from (i), x = 14 15 m = 210 m

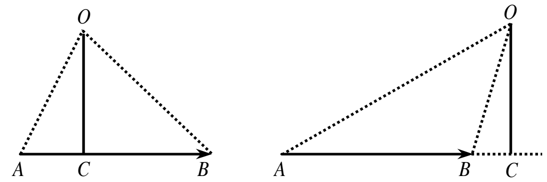
So, the cat will catch the mouse in 15 seconds after running a distance of 210 m, i.e., at a distance of 225 m from the starting point of the cat).

15. Define the moment of a force about a point and interpret its geometrical meaning. Prove that the algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point. [6]

Solution

Momentum of a force about a point: Let F be a force acting at point A on a body. The moment of the force about point O is defined as the vector product $\overrightarrow{OA} \quad \overrightarrow{F}$.

Geometrical interpretation:



Let force F acting on a body be represented by AB in magnitude and direction, and O the point about which the moment of force F is taken. Draw OC perpendicular to AB to meet the line of action of F at C. Also, draw OA and OB. By the construction, we have

Moment of F about O $= F \quad OC = AB \quad OC$

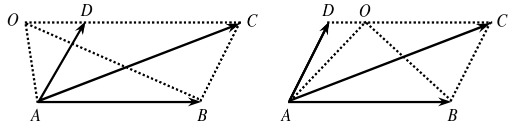
$= 2$ Area of triangle OAB.

Thus, geometrically **a moment of a force about a point is a vector normal to the plane containing the force and the force arm. Its magnitude is twice the area of the triangle formed by joining the point to the extremities of the line segment representing the force.**

<u>**Last part**</u>

Assume that forces P and Q meet at a point A.

Let O be the point about which moments are taken. Through O draw a line parallel to the line of action of P to meet the line of action of Q at D. Let AD represent Q in magnitude and direction and on the same scale AB represent P in magnitude and direction. Complete parallelogram $ABCD$. By parallelogram of forces, AC represents the resultant R of P and Q in magnitude and direction. Draw OA and OB.



(a) (b)

If, for example, by ì ΔOABî we mean "the area of ì ΔOABî, then clearly, 2 ΔOAB, 2 ΔOAD, 2 ΔOAC represent the moments of the forces P, Q, R, respectively, about O.

Hence, if O lies outside angle DAB as in fig (a), then the algebraic sum of the moments of forces P and Q about O is

$$2\Delta OAB + 2\Delta OAD = 2(\Delta ABC + \Delta OAD) = 2(\Delta ACD + \Delta OAD)$$
$$= 2\Delta OAC$$
$$= \text{Moment of resultant } R \text{ about } O$$

If O lies within angle DAB as in fig. (b), then the algebraic sum of the moments of forces P and Q about O is

$$2\Delta OAB \text{ ñ } 2\Delta OAD = 2(\Delta ABC \text{ ñ } \Delta OAD) = 2(\Delta ACD \text{ ñ } \Delta OAD)$$
$$= 2\Delta OAC$$
$$= \text{Moment of resultant } R \text{ about } O$$

Therefore, in either case, the algebraic sum of the moments of P and Q about O is equal to the moment of R about O.

Group ëCí

16. a) If a man rides his car at 25 km/hr, he has to spend Rs.2 per km on petrol. If he rides it at a faster speed of 40 km/hr, the petrol cost increases to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to find the maximum distance he can travel within one hour. Formulate the above problem as a linear programming problem. [2]

Solution

Let x and y be the distances covered at 25 km/hr and 40 km/hr respectively. Then total distance covered by $(x + y)$ km.

Case I

When the speed is 25 Km/Hr

Expenditure on petrol = Rs.2 per km

Hence for x km, Expenditure on petrol = Rs.$2x$

Case II

When the speed is 40 Km/Hr

Expenditure on petrol = Rs.5 per km

Hence for y km, Expenditure on petrol = Rs.$5y$

Therefore, total expenditure = Rs $(2x + 5y)$

But Amount to spend on petrol = Rs 100

$\therefore\ 2x + 5y \le 100$

Again, the time taken for x km = $\frac{x}{25}$

the time taken for y km = $\frac{y}{40}$

Therefore, Total time = $\frac{x}{25} + \frac{y}{40}$

$\therefore\ \frac{x}{25} + \frac{y}{40} \le 1$

$\Rightarrow\ 8x + 5y \le 200$

The mathematical model of the above problem is

Maximize: $F(x, y) = x + y$

subject to

$2x + 5y \le 100$

$8x + 5y \le 200$

$x \ge 0, y \ge 0.$

b) Convert the decimal number 2011 into octal form. [2]

Solution

Division	Reminder	
$\frac{2011}{8} = 251\frac{3}{8}$	3	LSD
$\frac{251}{8} = 31\frac{3}{8}$	3	
$\frac{31}{8} = 3\frac{7}{8}$	7	
$3\frac{3}{8}$	3	MSD

Check: $(2011)_{10} = (3733)$.

c) Is the following equations diagonally dominant:

12*x* + 3*y* ñ 5*z* = 1, *x* + 5*y* + 3*z* = 28, 3*x* + 7*y* + 13*z* = 1? [2]

Solution

For the first equation,

|diagonal element| = |coef. of *x*|

= |12| = 12 > 3 + 5 = |3| + |ñ5|.

For the second equation,

|diagonal element| = |coef. of *y*|

= |5| = 5 > 1 + 3 = |1| + |3|.

For the third equation,

|diagonal element| = |coef. of *z*|

= |13| = 13 > 3 + 7 = |3| + |7|.

Therefore, the given system of equations is diagonally dominant.

17. a) Using Gauss elimination method, solve the following system of equations:

$x + 3y - z = -2$

$3x + 2y - z = 3$

$-6x - 4y - 2z = 18$. [4]

OR

Solve the following equations using Gauss-Seidal method:

$2x_1 - x_2 = 8$

$3x_1 + 7x_2 = -5$. [4]

Solution

Given system is 3*x* + 3*y* ñ *z* = ñ2

3*x* + 2*y* ñ *z* = 3

ñ6*x* + 4*y* ñ 2*z* = 18.

Step 1: Forward elimination:

Subtracting 1st equation multiplied by 3 from 2nd equation and subtracting 1st equation multiplied by 2 from 3rd equation we get the system

x + 3*y* ñ *z* = ñ2

ñ 4*y* + 12*z* = ñ8

ñ 2*y* + 7*z* = ñ2

Dividing 2nd equation by ñ4 and adding the resulting 2nd equation multiplied by 2 from 3rd equation, we get

x + 3*y* ñ *z* = ñ2

y ñ 3*z* = 2

z = 2

∴ *z* = 2

Step 2: Backward substitution:

By the backward substitution, we get

y ñ 3 2 = 2

∴ *y* = 8

Also, *x* + 3 8 ñ 2 = ñ2

∴ *x* = ñ24

Hence the required solution is (*x*, *y*, *z*) = (ñ24, 8, 2).

<u>OR Part</u>

Given system is $2x_1 - x_2 = 8$

$3x_1 + 7x_2 = -5$

For the 1st equation,

|diagonal element| = |coeff. of x_1| = |2| > |ñ1| = |coeff. of x_2|

For the 2nd equation

|diagonal element| = |coeff. of x_2| = |7| > |3| = |coeff. of x_1|

Thus, the given system is diagonally dominant we apply Gauss-Siedel method as follows:

$x_1 = \frac{1}{2}(8 + x_2)$

$x_2 = \frac{1}{7}$ (ñ5 ñ $3x_1$)

Iteration I

Initially, start with estimation be $(x_{10}, y_{20}) = (0, 0)$. Then

$x_{11} = \frac{1}{2}(8 + 0) = 4$

$y_{21} = \frac{1}{7}$ (ñ5 ñ 3 4) = ñ2.4286

Iteration II

$x_{12} = \frac{1}{2}(8 + y_{21}) = \frac{1}{2}$ (8 ñ 2.4286) = 2.7857

$y_{22} = \frac{1}{7}$ (ñ5 ñ $3x_{12}$) $= \frac{1}{7}$ (ñ5 ñ 3 2.786) = ñ1.9081

Iteration III

$$x_{13} = \frac{1}{2}\ (8 + y_{22}) = \frac{1}{2}\ (8\ ñ\ 1.9081) = 3.0459$$

$$y_{23} = \frac{1}{7}\ (ñ5\ ñ\ 3x_{13}) = \frac{1}{7}\ (ñ5\ ñ\ 3\quad 2.7857) = ñ\ 2.0196$$

Iteration IV

$$x_{14} = \frac{1}{2}\ (8 + y_{23}) = \frac{1}{2}\ (8 + ñ2.0196) = 2.9001$$

$$y_{24} = \frac{1}{7}\ (ñ5\ ñ\ 3x_{14}) = \frac{1}{7}\ (ñ5\ ñ\ 3\quad 3.0459) = ñ1.9957$$

∴ 4th estimated solution is $(x_{14}, y_{24}) = (2.9001, ñ1.9957)$ which is close to exact solution $(x_1, x_2) = (3, ñ2)$.

b) Evaluate the following integral using Simpsonís rule:

$\int_0^1 \frac{dx}{1+x^2}$ **, taking 4 equal intervals (i.e. $n = 4$).** **[4]**

Solution

We have

$$\int_0^1 \frac{dx}{1+x^2}$$

where $n = 4$, $a = 1$, $b = 1$ and $f(x) = \frac{1}{1+x^2}$

Now,

$$h = \frac{b\ ñ\ a}{n} = \frac{1\ ñ\ 0}{4} = 0.25$$

Now, the results of computations are tabulated as follows:

i	x_i	$f(x_i)$
0	0	1
1	0.25	0.941
2	0.5	0.8
3	0.75	0.64
4	1	0.5

By Simpson's 1/3 rule we have

$$\int_0^1 \frac{dx}{1+x^2} \approx \frac{(b\ ñ\ a)}{3n}\ [f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)]$$

$$\approx \frac{1\ ñ\ 0}{3\quad 4}\ [1 + 4\quad 0.941 + 2\quad 0.8 + 4\quad 0.64 + 0.5]$$

$$\approx \frac{1}{12}\ [9.424]$$

$$\approx 0.785$$

18. Using Simplex method, maximize $Z = 5x_1 + 7x_2$ subject to:

$2x_1 + 3x_2 \leq 13$

$3x_1 + 2x_2 \leq 12$

$x_1, x_2 \geq 0$. **[6]**

Solution

In order to solve it by the simplex method we introduce slack variables s and t by setting

$s = 13\ ñ\ (2x_1 + 3x_2)$

$t = 12\ ñ\ (3x_1 + 2x_2)$.

Clearly, $s, t \geq 0$. Now the given LP problem can be restated in the **canonical form** as follows:

$2x_1 + 3x_2 + 1s + 0t + 0Z = 13$

$3x_1 + 2x_2 + 0s + 1t + 0Z = 12$

$ñ\ 5x_1\ ñ\ 7x_2 + 0s + 0t + 1Z = 0$.

This produces the following initial tableau:

Initial (First) tableau

Basic variables	x_1	x_2	s	t	Z	*RHS*
s	2	3	1	0	0	13
t	3	2	0	1	0	12
Z	-5	-7	0	0	1	0

The presence of negative numbers: ñ 7, ñ 5 in the last row indicates that the initial solution given by the initial tableau is not optimal. Since ñ 7 < ñ 5, the x_2-column is the **pivot column** and x_2 is the **entering variable**.

Dividing the last element in each row by the corresponding element in the pivot column, we obtain

13/3 = 4, 12/2 = 6.

Since 4 < 6, the second row is the **pivot row** and s is the **departing variable**. We see that 3 lies in the intersection of pivot column and pivot row. Dividing the pivot row by the **pivot element** 3, we obtain

Basic variables	x_1	x_2	s	t	Z	RHS
x_2	2/3	1	1/3	0	0	13/3
t	3	2	0	1	0	12
Z	-5	-7	0	0	1	0

Changing R_3 into R_3 ñ $2R_2$ and R_4 into $R_4 + 7R_2$, we obtain

Second tableau

Basic variables	x_1	x_2	s	t	Z	RHS
x_2	2/3	1	1/3	0	0	13/3
t	5/3	0	ñ2/3	1	0	10/3
Z	ñ1/3	0	7/3	0	1	91/3

The presence of negative number: ñ 1/3 in the last row indicates that the solution given by the second tableau is not optimal. Since there is only one negative number in the last row, the x_1-column is the **pivot column** and x_1 is the **entering variable**.

Dividing the last element in each row by the corresponding element in the pivot column, we obtain

(13/3)/(2/3) = 13/2, (10/3)/(5/3) = 2.

Since 2 < 13/2, the third row is the **pivot row** and t is the **departing variable**. We see that 5/3 lies in the intersection of pivot column and pivot row. Dividing the pivot row by the **pivot element** 5/3, we obtain

Basic variables	x_1	x_2	s	t	Z	RHS
x_2	2/3	1	1/3	0	0	13/3
x_1	1	0	ñ2/5	3/5	0	2
Z	ñ1/3	0	7/3	0	1	91/3

Changing R_2 into R_2 ñ $(2/3)R_3$ and R_4 into $R_4 + (1/3)R_2$, we obtain

Third tableau

Basic variables	x_1	x_2	s	t	Z	RHS
x_2	0	1	3/5	ñ6/5	0	3
x_1	1	0	ñ2/5	3/5	0	2
Z	0	0	7/3	1/5	1	31

As the bottom row contains no negative numbers, the third tableau gives the optimal solution. Hence the maximum value of Z is 31 at (2, 3).

19. Show that the equation $f(x) = x^3 - 18 = 0$ has only one positive root. Using bisection method, find the positive root correct to 3 places of decimal in the interval (2, 3). [6]

OR

Use Newton-Raphson method to find the positive root of $x^3 + 3x$ ñ $5 = 0$ lying between 1 and 2 correct to three places of decimals. [6]

Solution

We have

$f(x) = x^3$ ñ $18 = 0$

We apply the bisection method for the interval (2, 3). We get

$f(2) = 8$ ñ $18 =$ ñ10

$f(3) = 27$ ñ $18 = 9$.

Clearly, $f(2)$ $f(3) =$ ñ10 9 = ñ90 < 0.

Set $(a_0, b_0) = (2, 3)$, $x_0 = \frac{a_0 + b_0}{2} = \frac{2+3}{2} = 2.5$

Then $f(x_0) = f(2.5) =$ ñ2.375

On computation we obtain the following table of approximations.

n	x_n	$f(x_n)$
0	2.5	ñ2.375
1	2.75	2.796875
2	2.625	0.87891
3	2.5625	ñ1.17359
4	2.59375	ñ0.55045
5	2.609375	ñ0.23319
6	2.6171875	ñ0.07313
7	2.62109375	0.00726
8	2.619140625	ñ0.03296
9	2.6201171875	ñ0.01286
10	2.62060546875	ñ0.0028

From the table, $f(x_{10}) = f(2.62060546875) =$ ñ 0.0028 < 0.

On computation, we have $f(2.621) = 0.00533 > 0$.

Thus, by Bolzano's theorem the exact solution lies between 2.62060546875 and 2.621. Hence $x_{10} = 2.62060546875$ is an approximate solution correct up to 3 decimal places.

<u>OR Part</u>

We have

$f(x) = x^3 + 3x - 5$

$f'(x) = 3x^2 + 3$

$f(1) = 1 + 3 - 5 = -1$

$f(2) = 2^3 + 3 \cdot 2 - 5 = 9.$

Let initial approximation is 2. Then by Newton's method, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{9}{3 \quad 2^2 + 3} = 1.4.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.4 - \frac{(1.4)^3 + 3 \quad 1.4 - 5}{3 \quad (1.4)^2 + 3} = 1.181081.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.181081 - \frac{(1.181081)^3 + 3 \quad 1.181081 - 5}{3 \quad (1.181081)^2 + 3} = 1.154525.$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.154525 - \frac{(1.154525)^3 + 3 \quad 1.154525 - 5}{3 \quad (1.154525)^2 + 3} = 1.154171.$$

Hence the required root of given function is 1.154171 which is correct up to 3 decimal places.

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